Some Useful formulas: LIMIT OF SOME SPECIAL FUNCTIONS

(i) $\lim_{x\to\infty}\frac{1}{x}=0$ (ii) $\lim_{x\to-\infty}(1+\frac{1}{x})^x=e$ (iii) $\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$

(iv) $\lim_{x\to 0}\frac{\sin x}{x}=1=\lim_{x\to 0}\frac{\tan^{-1}x}{x}=\lim_{x\to 0}\frac{\sin^{-1}x}{x}=\lim_{x\to 0}\frac{\tan^{-1}x}{x}$

(v) $lim_{x\to\infty}\frac{e^x-1}{x}=1$ (vi) $lim_{x\to\infty}\frac{a^x-1}{x}=In\,a\ ,a>0$ (v) $lim_{x\to\infty}\frac{x^n-a^n}{x-a}=na^{n-1}$

INDETERMINATE FORMS $\frac{0}{0},\frac{\infty}{\infty},0\times\infty\ ,0^0\ ,\infty^0\ ,\infty-\infty\ ,1^\infty$ resolve indeterminate form before using the limit by using L-hospital rule or by solving the fractions.

DIFFERENTIAL AND INTEGRAL CALCULUS

First Principle: The *derivative* of the function *f(x)* is the function *f'(x)* defined by

$$f'(x)\equiv\frac{d}{dx}[f(x)]\equiv\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

S.No	Differentiation	Integration
1	$\frac{d}{dx}x^n=nx^{n-1}$	$\int x^n dx=\frac{x^{n+1}}{n+1}+c,$ n≠ -1
2	$\frac{d}{dx}e^{ax}=ae^{ax}$	$\int e^{ax}dx=\frac{e^{ax}}{a}$
3	$\frac{d}{dx}\log_e x=\frac{1}{x}$	$\int\frac{1}{x}dx=\log x$
4	$\frac{d}{dx}\log_a x=\frac{1}{x}\log_a e$	$\int a^x dx=\frac{a^x}{\log_e a}$
5	$\frac{d}{dx}\sin ax=a\cos ax$	$\int\sin ax\,dx=-\frac{\cos ax}{a}$
6	$\frac{d}{dx}\cos ax=-a\sin ax$	$\int\cos ax\,dx=\frac{\sin ax}{a}$
7	$\frac{d}{dx}\tan ax=a\sec^2 ax$	$\int\tan ax\,dx=\frac{-\log\sec ax}{a}=\frac{\log\cos ax}{a}$ $\int\sec^2 ax\,dx=\frac{\tan ax}{a}$
8	$\frac{d}{dx}\cot ax=-a\cos ec^2 ax$	$\int\cot ax\,dx=\frac{-\log\cos ecax}{a}=\frac{\log\sin ax}{a}$ $\int\cos ec^2 ax\,dx=\frac{-\cot ax}{a}$
9	$\frac{d}{dx}\sec ax\,dx=a\sec ax\tan ax$	$\int\sec ax\tan ax\,dx=\frac{\sec ax}{a}$ $\int\sec x\,dx=\log(\sec x+\tan x)=\log\ \tan(\frac{\pi}{4}+\frac{x}{2})$

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10	$\frac{d}{dx}\cos ec ax = -a \cos ec\, ax \cot ax$	$\int \cos ec ax.\cot ax\ dx = \frac{-\cot ax}{a}$ $\int \cos ec\, x\, dx = \log(\cos ec\, x - \cot x) = \log \tan \frac{x}{2}$
11	$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x$
12	$\frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}} dx = -\cos^{-1} x$
13	$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2} dx = \tan^{-1} x$
14	$\frac{d}{dx}\cot^{-1} x = \frac{-1}{1+x^2}$	$\int \frac{1}{1+x^2} dx = -\cot^{-1} x$
15	$\frac{d}{dx}\sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}} dx = \sec^{-1} x$
16	$\frac{d}{dx}\cos ec^{-1} x = -\frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}} dx = -\cos ec^{-1} x$
17	**MULTIPLICATION FORMULA** $\frac{d}{dx} f_1(x).f_2(x) = f_2(x).\frac{d}{dx} f_1(x) + f_1(x).\frac{d}{dx} f_2(x)$	**MULTIPLICATION FORMULA** $\int u.v\, dx = u\int v\, dx - \int \{\frac{d}{dx}u . \int v dx\} dx$
18	**DIVISION FORMULA(Quotient Rule)** $\frac{d}{dx}(\frac{f_1}{f_2}) = \frac{f_2.(\frac{d}{dx} f_1) - f_1.(\frac{d}{dx} f_2)}{(f_2)^2}$	**Leibnitz′successive integration by Parts** $= u\int v dx - u'\int\int v\, dx^2 + u''\int\int\int v dx^3 .... \underbrace{\int\int\int}_{n} v dx^n$
19	$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$	$\int \frac{1}{\sqrt{x}} dx = \int x^{-1/2} dx = \frac{x^{1/2}}{1/2}$

Some Other Formulae for Integration

$\int \frac{1}{\sqrt{a^2-x^2}} dx = \frac{1}{a}\sin^{-1}\frac{x}{a}$	$\int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$
$\int \frac{1}{a^2-x^2} dx = \frac{1}{2a}\log(\frac{a+x}{a-x}) = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)$, $-a < x < a$	
$\int \frac{1}{\sqrt{a^2+x^2}} dx = \log(x+\sqrt{a^2+x^2}) = \sinh^{-1}\left(\frac{x}{a}\right)$	$\int \frac{1}{\sqrt{x^2-a^2}} dx = \log(x+\sqrt{x^2-a^2}) = \cosh^{-1}\left(\frac{x}{a}\right)$
$\int \sqrt{a^2-x^2} dx = \frac{1}{2}[x\sqrt{a^2-x^2} + a^2\sin^{-1}\frac{x}{a}]$	
$\int \sqrt{x^2+a^2} dx = \frac{1}{2}[x\sqrt{x^2+a^2} + a^2\log(x+\sqrt{x^2+a^2})]$	$\int \sqrt{x^2-a^2} dx = \frac{1}{2}[x\sqrt{x^2-a^2} + a^2\log(x-\sqrt{x^2+a^2})]$

$\int e^{ax}.\sin bx\,dx = \frac{e^{ax}}{\sqrt{a^2+b^2}}[a\sin bx - b\cos bx]$	$\int e^{ax}.\cos bx\,dx = \frac{e^{ax}}{\sqrt{a^2+b^2}}[a\cos bx + b\sin bx]$

Differentiation of Hyperbolic Functions:

$f(x)$	$\sinh x$	$\cosh x$	$\tanh x$	$\text{sech}\,x$	$\text{cosech}\,x$	$\coth x$
$\frac{d}{dx}f(x)$	$\cosh x$	$\sinh x$	$\sec^2 h\,x$	$-\tanh x\,\text{sech}\,x$	$-\text{cosech}\,x\coth x$	$\text{cosech}^2\,x$

Note: the order of int. of the function is dependent on the nature of function. For convenience we use the following concept of

ILATE [I-INVERSE FUNCTION; L-LOGRITHEMIC FUNCTION; A-ALGEBRAIC FUNCTION T- TRIGONOMETTREC FUNCTION; E-EXPONENTIAL FUNCTION]

Definite Integral : $\int_0^\infty \frac{\sin x}{x}dx = \frac{\pi}{2}$, $\int_0^\infty \frac{\cos x}{x}dx = \infty$, $\int_0^\infty e^{-a^2x^2}dx = \frac{\sqrt{\pi}}{2a}$, $\int_{-\infty}^\infty e^{-a^2x^2}dx = \frac{\sqrt{\pi}}{a}$,

$\int_{-\infty}^\infty e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}$, $\int_0^\infty e^{-ax}dx = \frac{1}{a}$, $\int_0^\infty xe^{-x^2}dx = \frac{1}{2}$,

Fourier Transform

If $f(x)$ is the function defined in the interval $(-\infty, \infty)$, uniformly continuous in the finite intervals and $\int_{-\infty}^\infty |f(x)|dx$ converges then the Fourier transform of a one-dimensional function f(x) is defined as

$$\Im[f(x)] = F(s) = \frac{1}{2\pi}\int_{-\infty}^\infty f(x)e^{isx}\,dx$$ [Note: One can leave coefficient 1/2π] .

The inverse transform $\Im^{-1}$ is defined as

$$\Im^{-1}[F(s)] = f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty F(s)e^{-isx}\,ds$$, Where s is a parameter.

It may be represented by

$$\Im[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(x)e^{isx}\,dx$$

and

$$\Im^{-1}[F(s)] = f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty F(s)e^{-isx}\,ds$$

and $\Im[f(x)] = F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty f(x)e^{2\pi isx}\,dx$ and $\Im^{-1}[F(s)] = f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^\infty F(s)e^{-2\pi isx}\,ds$

Remark: 1. Since $|x| = \begin{cases} x & , x>0 \\ -x & , x<0 \end{cases}$ hence means $x > a$ and $-x < a$ $\Rightarrow x > a$ and $x > -a$ or $-a<x<a$

2. $|x| > a$ means $-\infty.........-a..........0.............a............\infty$: $a< x < \infty$ and $-\infty < x < -a$

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Thus $f(x) = \begin{cases} x, & |x| \le a \\ 0, & |x| \ge a \end{cases} = \begin{cases} x, & -a < x < a \\ 0, & -\infty < x < -a \text{ and } a < x < \infty \end{cases}$

- **For evaluation of integrals we use inverse Fourier transform.**

Fourier Transform (o r Fourier Complex Transform):

Q. 1 . Write Linear and Change of scale property for Fourier Transform. **[June 2014]**

Q. 2 . State and prove shifting property for Fourier Transform. **[Hint:** $F\{f(x-a)\} = e^{isa} f(s)$]

Q. 3 . State and Prove **Convolution Theorem** for Fourier Transform.

Q. 4 . Find the Fourier complex transform of *f(x)* ,if $f(x) = \begin{cases} e^{iwx}, & a < x < b \\ 0, & x < a, \; x > b \end{cases}$

[Ans: $F\{f(x)\} = -\frac{i}{s+w}[e^{i(s+w)b} - e^{i(s+w)a}]$**]**

Q. 5 . Find the Fourier complex transform of *f(x)* ,if $f(x) = \begin{cases} \frac{\sqrt{2\pi}}{2\varepsilon}, & |x| \le \varepsilon \\ 0, & |x| > \varepsilon \end{cases}$ **[Ans:** $F\{f(x)\} = \frac{\sin s\varepsilon}{s\varepsilon}$**]**

Q. 6 . Find the **Fourier transform** of $f(x) = \begin{cases} x, & |x| \le a \\ 0, & |x| > a \end{cases}$

Q. 7 . Find the Fourier transform of $f(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$ Hence evaluate

(i) $\int_0^\infty \frac{\sin sa \cos sx}{s} ds$ (ii) $\int_0^\infty \frac{\sin s}{s} ds$ (iii) $\int_0^\infty \frac{\sin x}{x} dx$ **[2003, June 17]** **[Ans: (i)** $\frac{\pi}{2}$ **(ii)** $\frac{\pi}{2}$ **(iii)** $\frac{\pi}{2}$**]**

Q. 8 . Find the **Fourier transform** of $f(x) = \begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$ Hence evaluate $\int_0^\infty \frac{x\cos x - \sin x}{x^3} \cos\frac{x}{2} dx$

Q. 9 . Find Fourier Transform of **Dirac Delta Function**.

[Hint : Dirac Delta function $\delta(t-a) = \lim_{h\to 0} I(h, t-a) = \lim_{h\to 0} \begin{cases} \frac{1}{h}, & a < t < a+h \\ 0, & t < a, t > a+h \end{cases}$ Ans: $\frac{e^{isa}}{\sqrt{2\pi}}$]

Q. 10 . Find the Fourier **transform** of the function , $f(x) = e^{-ax^2}$, $a > 0$ **[June 2015, 17]**

Q. 11 . Show that the **Fourier transforms** of $f(x) = e^{-\frac{x^2}{2}}$ is self reciprocal.

Q. 7 . Find Fourier Transform of $f(x) = e^{-|x|}$ (or . $f(x) = e^{-a|x|}$). [Ans : $f(x) = \sqrt{\frac{2}{\pi}} \frac{a}{a^2+s^2}$]

Fourier Sine Transform:

$$\Im_s\left[f(x)\right]=F_s(s)=\int_0^\infty f(x)\sin sx\,dx=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin sx\,dx$$

and its inverse transform $\Im^{-1}{}_s\left[F(s)\right]=f_s(x)=\int_0^\infty F(s)\sin sx\,ds=\sqrt{\frac{2}{\pi}}\int_0^\infty F(s)\sin sx\,ds$

Fourier Cosine Transform:

$$\Im_c\left[f(x)\right]=F_c(s)=\int_0^\infty f(x)\cos sx\,dx=\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos sx\,dx=\frac{1}{\sqrt{2\pi}}\int_0^\infty f(x)\cos sx\,dx$$

And its inverse transform

$$\Im^{-1}{}_c\left[F(s)\right]=f_c(x)=\int_0^\infty F(s)\cos sx\,ds=\sqrt{\frac{2}{\pi}}\int_0^\infty F(s)\cos sx\,ds=\frac{1}{\sqrt{2\pi}}\int_0^\infty F(s)\cos sx\,ds$$

Fourier Sine and Cosine Transform:

Q. 8 . Find the Fourier sine transform of $f(x)=e^{-3x}+e^{-4x}$ **[June 17]**

Q. 9 . Find the **cosine transform** of the function $f(x)=\begin{cases}\cos x & ;\ 0<x<a\\ 0 & ;\ x>a\end{cases}$

Q. 10 . Find the **sine transform** of the function , $f(x)=\begin{cases}\sin x & ;\ 0<x<a\\ 0 & ;\ x>a\end{cases}$ **[Dec. 2014]**

Q. 11 . Find Fourier sine and cosine transform of e^{-x} and recover the original function using inverse formula**.**

Q. 12 . Using Fourier integral show that $\int_0^\infty \frac{\lambda\sin\lambda x}{\lambda^2+k^2}d\lambda=\frac{\pi}{2}e^{-kx}$, $k>0, x>0$ **[Hint:** Find F.Sine T. of $f(x)=e^{-kx}$**]**

Q. 13 . Find the **sine and cosine transform** of the function , $f(x)=e^{-ax}$

Q. 14 . Find the cosine transform of the function $f(x)=e^{-|x|}, x\geq 0$ and prove that $\int_0^\infty \frac{\cos sx}{1+s^2}ds=\frac{\pi}{2}e^{-x}$

Q. 15 . Find the Fourier sine transform of the function $f(x)=e^{-|x|}, x\geq 0$ and prove that

$$\int_0^\infty \frac{x\sin mx}{1+x^2}dx=\frac{\pi}{2}e^{-m}, m>0$$

[June2014]

Q. 16 . Find the Fourier transform of $f(x)=\begin{cases}x & ;0<x<1\\ 2-x & ;1<x<2\\ 0 & ;x>2\end{cases}$ **[June 17]**

Q. 17 . Find Fourier Sine and Cosine Transform of $f(x) = e^{-|x|}$. Hence evaluate $\int_0^\infty \frac{x \sin mx}{1+x^2} dx$ **.[Dec. 2011]**

Q. 18 . Prove that (i) $F_s\{xf(x)\} = -\frac{d}{ds} F_c\{f(x)\}$ (ii) $F_c\{xf(x)\} = \frac{d}{ds} F_s\{f(x)\}$. Hence evaluate Fourier cosine and sine transform of $f(x) = xe^{-ax}$.

Fourier transforms using Differentiation:

Q. 19 . Find the **sine transform** of the function , $f(x) = \frac{1}{x}$

Q. 20 . Find the Fourier sine transform of $f(x) = \frac{e^{-ax}}{x}$. Hence find the Fourier sine transform of $1/x$.

[June 2016,17]

Q. 21 . Find Fourier cosine transform of $f(x) = \frac{1}{1+x^2}$ and hence derive fourier sine transform of $f(x) = \frac{x}{1+x^2}$ [**Hint** Find F. Cosine T. and Diff. two time to make second order diff eq. and solve it]

Q. 22 . Find the Fourier sine(and Cosine) transform of $f(x) = e^{-x^2}$. Hence find the Fourier sine transform of *1/x*. [**Hint** Find F. Sine / Cosine T. and Diff. the eq. to make first order diff eq. and solve it]
